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A description of the high-density electron system in terms of bosons

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Abstract. Tomonaga's idea of describing the one-dimensional jellium model in terms of bosons is adopted for the three-dimensional case, but worked out in a completely different way. An algorithm is given for the construction of a boson Hamiltonian that takes into account the full dynamics of the jellium model at all densities.

The algorithm is applied to a reduced form of the jellium model, which has the same ground state energy as the jellium model in the high-density limit. The resulting boson Hamiltonian is compared with Sawada's Hamiltonian, which also has this ground state energy in the limit of high density. Finally, the present boson formulation is discussed briefly.

1. Introduction

The system of interacting electrons moving against a uniform background of neutralizing positive charge is known as the jellium model. Until now the ground state properties of the jellium model can only be calculated approximately except in the limiting cases of extremely high and low densities.

Several ways of approach exist for dealing with the electron system. One of them is to reformulate this system in terms of bosons. The boson formulation of the one-dimensional model has been discussed by Tomonaga [1]. His formulation holds in the high-density limit, as then only scattering processes with small momentum transfer are important. The three-dimensional system has been discussed by Sawada in terms of a free boson Hamiltonian [2]. His purpose was the justification of the ground state energy calculation by Gell-Mann and Brueckner [3], who summed an apparently divergent series of ring diagrams. That calculation, which is equivalent to the random phase approximation of Bohm and Pines [4], leads to the exact ground state energy in the high-density limit. Sawada's Hamiltonian is the result of discarding all interaction terms that do not generate ring diagrams. Consequently Sawada's boson description is only equivalent to the fermion description within the framework of perturbation theory. This means that Sawada's three-dimensional approach is not analogous to Tomanaga's formulation, as the latter one takes into account the full dynamics of the one-dimensional system in the high-density limit. The results of Arponen and Pajanne [5], who generalized Sawada's Hamiltonian and obtained a boson Hamiltonian, which is non-Hermitian and of sixth order in the boson operators, support this conclusion.

The purpose of the present paper is to obtain a Tomonaga-like boson formulation of the three-dimensional fermion system. In section 2 the algorithm is given for the construction of a boson Hamiltonian H_B, that takes into account the full dynamics of the jellium model at all densities. An explicit construction of H_B appears impossible in general. In section 3

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the given algorithm is applied to a reduced form of the jellium model, that allows an explicit construction and has the same ground state energy as the jellium model in the high-density limit. The resulting boson Hamiltonian is compared with Sawada's Hamiltonian. Unlike Sawada's expression the present one describes a fermion system and can therefore be considered as a three-dimensional analogue of Tomonaga's Hamiltonian. The significance of the present boson formulation is discussed in section 4. Finally it should be remarked that the present paper can be considered as a logical continuation of a previous treatise on the boson formulation of the low-density electron system [6].

2. The boson formulation

The plane wave representation of the jellium model is given by the Hamiltonian

$$H = \sum_{k\sigma} \epsilon_k c^+_{k\sigma} c_{k\sigma} + \frac{1}{2} {\sum_{\substack{q,k,k' \\ \sigma,\sigma'}}}' V(q) c^+_{k\sigma} c^+_{k'\sigma'} c_{k'+q\sigma'} c_{k-q\sigma} \tag{2.1}$$

where

$$V(q) = \frac{4\pi e^2}{\Omega q^2} \tag{2.2}$$

and

$$\epsilon_k = \frac{\hbar^2 k^2}{2m}. \tag{2.3}$$

Here e and m are the charge and mass of an electron, respectively, and Ω denotes the volume of the system, that is thought to consist of $2N$ electrons. The operators $c_{k\sigma}$ and $c^+_{k\sigma}$ describe the annihilation and creation of a fermion having wavevector k and spin σ, respectively. The prime appearing in the summation over the momentum transfers q indicates that the $q = 0$ term is excluded in consequence of the homogeneous positively charged background.

The first step of the present boson formulation for the jellium model concerns the calculation of the matrix elements of the jellium model using the complete set of eigenstates of the kinetic energy operator appearing in (2.1). An eigenstate $|m\rangle$ of this set can be expressed as

$$|m\rangle = \left(\prod_{i=1}^{m} \theta(k_F - |k_i|)\theta(|k_i + q_i| - k_F) c^+_{k_i+q_i\tau_i} c_{k_i\sigma_i} \right) |0\rangle \tag{2.4}$$

where $|0\rangle$, the filled Fermi sphere, is the ground state of the kinetic energy operator and k_F denotes the radius of the Fermi sphere. The matrix elements can be easily calculated. The diagonal element $\langle m|H|m\rangle$ is given by

$$\langle m|H|m\rangle = \sum_{k\sigma} \left[\epsilon_k - \frac{1}{2} {\sum_{q}}' V(q) \langle m|n_{k+q\sigma}|m\rangle \right] \langle m|n_{k\sigma}|m\rangle \tag{2.5}$$

where

$$\langle m|n_{k\sigma}|m\rangle = \langle m|c^+_{k\sigma} c_{k\sigma}|m\rangle = \begin{cases} 1 & \text{if } |k| < k_F \text{ and } (k,\sigma) \neq (k_i,\sigma_i) \\ 1 & \text{if } |k| > k_F \text{ and } (k,\sigma) = (k_i + q_i, \tau_i) \\ 0 & \text{otherwise .} \end{cases} \tag{2.6}$$

The off-diagonal element $\langle m|H|m'\rangle$ is zero unless $|m'\rangle$ is of the following form:

$$|m'\rangle = c^+_{\ell_4\tau_2'}c_{\ell_2\tau_2}\ \ c^+_{\ell_3\tau_1'}c_{\ell_1\tau_1}|m\rangle \tag{2.7}$$

where the creation and annihilation operators refer to four mutually different one-electron states. This is a direct consequence of the two-body nature of the interaction term in (2.1). The non-zero matrix element $\langle m|H|m'\rangle$ is given by

$$\langle m|H|m'\rangle = [V(\ell_1 - \ell_3)\delta_{\tau_1\tau_1'}\delta_{\tau_2\tau_2'} - V(\ell_2 - \ell_3)\delta_{\tau_2\tau_1'}\delta_{\tau_1\tau_2'}]\delta_{\ell_1+\ell_2,\ell_3+\ell_4}. \tag{2.8}$$

Next the matrix elements are used as the guiding principle for the formulation of the jellium model in terms of a boson Hamiltonian H_B. In order to construct H_B, each fermion state $|m\rangle$ is replaced by a corresponding boson state $|\varphi_{Am}\rangle$. The set of boson states that corresponds with the complete set of fermion states is obtained by replacing each electron–hole pair $c^+_{k+q\sigma'}c_{k\sigma}$ with $|k+q| > k_F$ and $|k| < k_F$ by the corresponding boson operator $d^+_{q\sigma'}(k\sigma)$. The boson operators satisfy the commutation relations

$$\begin{aligned}&[d^+_{q\sigma'}(k\sigma), d^+_{q'\tau'}(k'\tau)] = 0\\ &[d_{q\sigma'}(k\sigma), d^+_{q'\tau'}(k'\tau)] = \delta_{qq'}\delta_{kk'}\delta_{\sigma\tau}\delta_{\sigma'\tau'}.\end{aligned} \tag{2.9}$$

The replacement means that the fermion state $|m\rangle$ (2.4) corresponds with the boson state

$$|\varphi_{Am}\rangle = \left(\prod_{i=1}^{m}\theta(k_F - |k_i|)\theta(|k_i + q_i| - k_F)d^+_{q_i\tau_i}(k_i\sigma_i)\right)|\varphi_{A0}\rangle \tag{2.10}$$

where $|\varphi_{A0}\rangle$ is the vacuum state for the bosons, i.e.

$$d_{q\sigma'}(k\sigma)|\varphi_{A0}\rangle = 0 \tag{2.11}$$

for all q, k, σ' and σ. Note that the restrictions $|k_i| < k_F$ and $|k_i + q_i| > k_F$ in (2.10) are superfluous, as the operators $d^+_{q\sigma'}(k\sigma)$ are defined for $|k| < k_F$ and $|k+q| > k_F$ only. Consequently they will no longer be mentioned explicitly.

Clearly the number of boson states that can be constructed in terms of the boson creation operator $d^+_{q\sigma'}(k\sigma)$ and the vacuum state $|\varphi_{A0}\rangle$ exceeds by far the number of fermion states. For instance a boson state like

$$|\varphi_{C2}\rangle = d^+_{q_1\tau_1}(k_1\sigma_1)d^+_{q_2\tau_2}(k_1\sigma_1)|\varphi_{A0}\rangle \tag{2.12}$$

does not correspond with any fermion state, whereas a boson state like

$$|\varphi_{B2}\rangle = -d^+_{q_1+k_1-k_2\tau_1}(k_2\sigma_2)d^+_{q_2+k_2-k_1\tau_2}(k_1\sigma_1)|\varphi_{A0}\rangle \tag{2.13}$$

corresponds with the same fermion state as the boson state $|\varphi_{A2}\rangle$ defined in (2.10). For clarity the complete set of boson states is divided up into three subsets A, B and C. Subset A consists of states $|\varphi_{Am}\rangle$ being in a one-to-one correspondence with the complete set of fermion states $|m\rangle$ (2.4). Subset B consists of those boson states $|\varphi_{Bm}\rangle$ that correspond with fermion states already taken into account by the states $|\varphi_{Am}\rangle$ (cf. 2.13). Finally subset C consists of boson states which do not correspond with any fermion state (cf. 2.12).

Now the boson Hamiltonian H_B is constructed by requiring

$$\begin{aligned}&\langle\varphi_{Am}|H_B|\varphi_{Am}\rangle = \langle m|H|m\rangle\\ &\langle\varphi_{Am}|H_B|\varphi_{Am'}\rangle = \langle m|H|m'\rangle\end{aligned} \tag{2.14}$$

for all fermion states $|m\rangle$ and $|m'\rangle$, and further

$$\langle\varphi_{Am}|H_B|\varphi_{Bm'}\rangle = \langle\varphi_{Am}|H_B|\varphi_{Cm'}\rangle = 0 \tag{2.15}$$

for all boson states of the respective subsets. The correctness of this procedure follows immediately from the form of the matrix representation of H_B, $\mathbf{M}(H_B)$, on the complete set of boson states:

$$\mathbf{M}(H_B) = \begin{pmatrix} \mathbf{M}_A & 0 \\ 0 & \mathbf{M}_{BC} \end{pmatrix} \tag{2.16}$$

where $\mathbf{M}_A$ is the matrix representation of H_B on the subset A, i.e. $\mathbf{M}_A$ is identical to the matrix representation of the jellium model, and $\mathbf{M}_{BC}$ is the matrix representation of H_B on the subset of the remaining boson states. Consequentely the boson system as described by H_B is not equivalent with the jellium model but includes that model. The merit of the present formulation is the separation of fermion and non-fermion states. This means that the eigenvalues of the jellium model form a subset of all eigenvalues of H_B. In this sense H_B can be said to describe the jellium model.

For constructional purposes H_B is written as

$$H_B = H_B^1 + H_B^2 + \Delta H_B \tag{2.17}$$

where H_B^1, H_B^2 and ΔH_B are chosen such that

$$\begin{aligned} &\langle\varphi_{Am}|H_B|\varphi_{Am}\rangle = \langle\varphi_{Am}|H_B^1|\varphi_{Am}\rangle \\ &\langle\varphi_{Am}|H_B|\varphi_{Am'}\rangle = \langle\varphi_{Am}|H_B^2|\varphi_{Am'}\rangle \qquad m \neq m' \\ &\langle\varphi_{Am}|H_B^1 + H_B^2|\varphi_{\mu m'}\rangle = -\langle\varphi_{Am}|\Delta H_B|\varphi_{\mu m'}\rangle \qquad \mu = B, C. \end{aligned} \tag{2.18}$$

The construction of the terms H_B^1 and H_B^2 is given in appendix A. The remaining terms ΔH_B, however, cannot be constructed explicitly in general. The reason is a lack of criterion for deciding whether a particular boson state belongs to subset A or B in the lower density case when interaction terms with large momentum transfer q are important.

3. The high-density limit

The boson Hamiltonian H_B, which includes the full description of the jellium model at all densities, cannot be constructed explicitly due to the complexity of ΔH_B as was mentioned in the previous section. The algorithm given in section 2, however, can be applied to simpler fermion systems, which allow an explicit description in terms of bosons. Here the algorithm of section 2 is applied to a reduced form H_F of the jellium model, which has the same ground state energy as the jellium model in the high-density limit.

The starting point of the present analysis is the following fermion Hamiltonian:

$$\begin{aligned} \tilde{H}_F = &\sum_{k,\sigma} \varepsilon_k c_{k\sigma}^+ c_{k\sigma} - \frac{1}{2} \sum_{k,q,\sigma}{}' V(q)\theta(k_F - k)\theta(k_F - |k+q|) \\ &+ \frac{1}{2} \sum_{q,k,k',\sigma,\sigma'}{}' \theta(k_F - k)\theta(k_F - k')\theta(|k+q| - k_F)\theta(|k'+q| - k_F)V(q) \\ &\times \left[(c_{-k'-q\sigma'}^+ c_{-k'\sigma'} + c_{k'\sigma'}^+ c_{k'+q\sigma'})(c_{k+q\sigma}^+ c_{k\sigma} + c_{-k\sigma}^+ c_{-k-q\sigma})\right]. \end{aligned} \tag{3.1}$$

As shown by Sawada [2], $\tilde{H}_F$ has the ground state energy of the jellium model in the high-density limit [3], namely

$$E_0 = 2N\left[\frac{2.2}{r_s^2} - \frac{0.9}{r_s} + 0.0622 \ln r_s\right] \tag{3.2}$$

where E_0 is given in Rydberg and the dimensionless parameter r_s is given by

$$r_s = \left(\frac{3\Omega}{8\pi N}\right)^{1/3} \frac{me^2}{\hbar^2}. \tag{3.3}$$

Sawada's Hamiltonian H_S is obtained by simply replacing the fermion pair operator $c^+_{k\sigma}c_{k\sigma}$ by the boson operator $n^B_{k\sigma}$ (A.1) and the electron–hole pair $c^+_{k+q\sigma}c_{k\sigma}$ with $|k+q| > k_F$ and $k < k_F$ by the boson operator $d^+_{q\sigma}(k\sigma)$, i.e.

$$H_S = \sum_{k,\sigma}{}' \varepsilon_k n^B_{k\sigma} - \frac{1}{2}\sum_{k,q,\sigma} V(q)\theta(k_F - k)\theta(k_F - |k+q|) + \frac{1}{2}\sum_{k,k',q,\sigma,\sigma'}{}' V(q)[d^+_{-q\sigma'}(-k'\sigma') + d_{q\sigma'}(k'\sigma')][d^+_{q\sigma}(k\sigma) + d_{-q\sigma}(-k\sigma)] \tag{3.4}$$

Clearly H_S does not describe a fermion system, as the eigenstates of H_S are linear combinations of states belonging to the subsets A, B and C.

The fermion Hamiltonian $\tilde{H}_F$ (3.1) still contains too many terms for the present purpose of describing the fermion system (3.1) in terms of bosons. Notably interaction terms with large momentum transfer $q \gtrsim k_F$, that do not contribute to the ground state energy in the high-density limit (3.2) as shown in appendix B, prevent an explicit description of the fermion system in terms of bosons. For that reason the algorithm of section 2 is not applied to $\tilde{H}_F$ itself but to a reduced form of $\tilde{H}_F$ leaving out interaction terms with $q \gtrsim k_F$. This reduced form is the following fermion Hamiltonian:

$$H_F = \sum_{k,\sigma}{}' \epsilon_k c^+_{k\sigma}c_{k\sigma} - \frac{1}{2}\sum_{k,q,\sigma} V(q)\theta(k_F - k)\theta(k_F - |k+q|) + \frac{1}{2}\sum_{q,k,k',\sigma,\sigma'}{}'' \theta(k_F - k)\theta(k_F - k')\theta(|k+q| - k_F)\theta(|k'+q| - k_F)V(q) \times \left[(c^+_{-k'-q\sigma'}c_{-k'\sigma'} + c^+_{k'\sigma'}c_{k'+q\sigma'})(c^+_{k+q\sigma}c_{k\sigma} + c^+_{-k\sigma}c_{-k-q\sigma})\right] \tag{3.5}$$

where the double prime appearing in the second summation over q indicates that this summation is restricted to those terms satisfying $0 < q < r_s^{1/4}k_F < |q+k+k'|$. The restriction $q < r_s^{1/4}k_F < |q+k'+k|$ must be imposed in order to obtain an explicit description of H_F in terms of bosons, as shown in the following. The influence of this restriction on the ground state energy in the high-density limit ($r_s \to 0$) is negligible (see appendix B).

The boson Hamiltonian H_{FB}, which describes the fermion system defined by (3.5), is now obtained from the fermion Hamiltonian H_F by requiring

$$\begin{aligned} \langle\varphi_{Am}|H_{FB}|\varphi_{Am'}\rangle &= \langle m|H_F|m'\rangle \\ \langle\varphi_{Am}|H_{FB}|\varphi_{Bm'}\rangle &= \langle\varphi_{Am}|H_{FB}|\varphi_{Cm'}\rangle = 0 \end{aligned} \tag{3.6}$$

for all fermion states $|m\rangle$ and all boson states $|\varphi_{Am}\rangle, |\varphi_{Bm}\rangle, |\varphi_{Cm}\rangle$. For constructional purposes H_{FB} is written as

$$H_{\mathrm{FB}} = H_{\mathrm{FB}}(0) + \Delta H_{\mathrm{FB}}(1) + \Delta H_{\mathrm{FB}}(2) \tag{3.7}$$

where $H_{\mathrm{FB}}(0)$, $\Delta H_{\mathrm{FB}}(1)$ and $\Delta H_{\mathrm{FB}}(2)$ must satisfy:

$$\begin{aligned}
&\langle\varphi_{Am}|H_{\mathrm{FB}}|\varphi_{Am'}\rangle = \langle\varphi_{Am}|H_{\mathrm{FB}}(0)|\varphi_{Am'}\rangle \\
&\langle\varphi_{Am}|H_{\mathrm{FB}}(0) + \Delta H_{\mathrm{FB}}(1)|\varphi_{Cm'}\rangle = \langle\varphi_{Am}|\Delta H_{\mathrm{FB}}(2)|\varphi_{Cm'}\rangle = 0 \\
&\langle\varphi_{Am}|H_{\mathrm{FB}}(0) + \Delta H_{\mathrm{FB}}(2)|\varphi_{Bm'}\rangle = \langle\varphi_{Am}|\Delta H_{\mathrm{FB}}(1)|\varphi_{Bm'}\rangle = 0.
\end{aligned} \tag{3.8}$$

The construction of $H_{\mathrm{FB}}(0)$ is straightforward and gives (cf. (A.1) and (A.15)–(A.17))

$$\begin{aligned}
H_{\mathrm{FB}}(0) = &\sum_{k,\sigma} \epsilon_k n^{\mathrm{B}}_{k\sigma} - \frac{1}{2}\sum_{k,q,\sigma}{}' V(q)\theta(k_{\mathrm{F}} - k)\theta(k_{\mathrm{F}} - |k+q|) \\
&+ \frac{1}{2}\sum_{q,k,k',\sigma,\sigma'}{}'' \{V(q)[d^+_{-q\sigma'}(-k'\sigma') + d_{q\sigma'}(k'\sigma')]\,[d^+_{q\sigma}(k\sigma) + d_{-q\sigma}(-k\sigma)] \\
&- V(q+k+k')[d^+_{-q\sigma'}(-k'\sigma)d^+_{q\sigma}(k\sigma') + d_{q\sigma}(k\sigma')d_{-q\sigma'}(-k'\sigma)] \\
&- 2V(k-k')d^+_{q\sigma}(k'\sigma')d_{q\sigma}(k\sigma')\,\}.
\end{aligned} \tag{3.9}$$

The only non-zero matrix elements of the type $\langle\varphi_{Am}|H_{FB}(0)|\varphi_{Cm'}\rangle$ are

(1) those between the states

$$|\varphi_{Am}\rangle = d^+_{q_1\sigma_1}(k_1\sigma_1')|\varphi_{Am-1}\rangle \qquad |\varphi_{Cm'}\rangle = d^+_{q_3\sigma_3}(k_3\sigma_3')d^{(+)}_{q_4\sigma_4}(k_4\sigma_4')|\varphi_{Am}\rangle$$

where $(k_3\sigma_3') = (k_1\sigma_1')$ or $(q_3\sigma_3) = (q_1 + k_1 - k_3\sigma_1)$ and

(2) those between the states

$$|\varphi_{Am}\rangle = d^+_{q_1\sigma}(k_1\sigma_1')d^+_{q_2\sigma_2}(k_2\sigma_2')|\varphi_{Am-2}\rangle \qquad |\varphi_{Cm'}\rangle = d^+_{q_3\sigma_3}(k_3\sigma_3')d^+_{q_4\sigma_4}(k_4\sigma_4')|\varphi_{Am}\rangle$$

where $(k_3\sigma_3') = (k_1\sigma_1')$ and $(k_4\sigma_4') = (k_2\sigma_2)$ or $(q_3\sigma_3) = (q_1 + k_1 - k_3\sigma_1)$ and $(k_4\sigma_4') = (k_2\sigma_2)$ or $(q_3\sigma_3) = (q_1 + k_1 - k_3\sigma_1)$ and $(q_4\sigma_4) = (q_2 + k_2 - k_4\sigma_2)$.

Consequently $\Delta H_{\mathrm{FB}}(1)$, which must compensate these matrix elements, is found to be:

$$\begin{aligned}
\Delta H_{\mathrm{FB}}(1) = -\sum_{q,k,k',\sigma,\sigma'}{}' \{&[N(q,k,\sigma,\sigma) \\
&- \tfrac{1}{2}N(q,k,\sigma,\sigma)N(-q,-k',\sigma',\sigma')]V(q)d_{q\sigma}(k\sigma)d_{-q\rho'}(-k'\rho') \\
&- [N(q,k,\sigma,\sigma') - \tfrac{1}{2}N(q,k,\sigma,\sigma')N(-q,-k',\sigma',\sigma)] \\
&\times V(q+k+k')d_{q\sigma'}(k\sigma)d_{-q\sigma}(-k'\sigma') \\
&+ N(q,k,\sigma,\sigma)V(q)d_{q\sigma}(k\sigma)d^+_{q\sigma'}(k'\sigma') \\
&- N(q,k,\sigma,\sigma')V(k-k')d_{q\sigma'}(k\sigma)d^+_{q\sigma'}(k'\sigma)\} + \mathrm{cc}
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
N(q_1,k_1,\sigma_1,\tau_1) = &-d^+_{q_1\tau_1}(k_1\sigma_1)d_{q_1\tau_1}(k_1\sigma_1) \\
&+ \sum_{q',\tau}[d^+_{q'\tau}(k_1\sigma_1)d_{q'\tau}(k_1\sigma_1) + d^+_{q_1+k_1-q'\tau_1}(q'\tau)d_{q_1+k_1-q'\tau_1}(q'\tau)].
\end{aligned}$$

The effect of $\Delta H_{FB}(2)$, which must compensate matrix elements of the type $\langle\varphi_{Am}|H_{FB}(0)|\varphi_{Bm'}\rangle$, is an additional restriction on the summations in (3.9). This restriction can be phrased as follows: taking into account the term proportional to

$$d^{(+)}_{-q\sigma'}(-k'\sigma')d^{(+)}_{q\sigma}(k\sigma)$$

means discarding the term proportional to

$$d^{(+)}_{q+k+k'\sigma}(-k'\sigma')d^{(+)}_{-q-k-k'\sigma'}(k\sigma).$$

Such a restriction cannot be made mathematically explicit in general. In the present case, however, this problem does not apply as the restriction due to $\Delta H_{FB}(2)$ is already contained in the relatively simple restriction $q < r_s^{1/4}k_F < |q+k+k'|$, i.e.

$$\Delta H_{FB}(2) = 0. \tag{3.11}$$

It should be remarked here that (3.11) does not hold at lower densities, where interaction terms with momentum transfer $q \gtrsim k_F$ become important for the ground state energy and a restriction $q < r_s^{1/4}k_F < |q+k+k'|$ cannot be imposed.

The Hamiltonian H_{FB}, given by (3.7), (3.9), (3.10) and (3.11), includes the full description of a reduced form of the jellium model with the same ground state energy as the jellium model in the high-density limit. In this sense H_{FB} can be considered as a three-dimensional analogue of Tomonaga's Hamiltonian for the one-dimensional electron system.

It is interesting to compare H_{FB} with Sawada's Hamiltonian H_S (3.4), which also leads to the exact ground state energy of the jellium model for $r_s \to 0$. The two Hamiltonians differ strongly from a fundamental point of view. The present Hamiltonian H_{FB} is a boson formulation of the fermion system defined by (3.5) whereas H_S does not include a description of a fermion system at all, i.e. its meaning is unclear.

4. Conclusions

The purpose of the present paper is to describe the interacting electron system in terms of bosons. The present boson formulation consists of an algorithm for the construction of a boson Hamiltonian H_B, which includes a full description of the jellium model at all densities. An explicit construction has been presented for a reduced form of the jellium model with the same ground state energy as the jellium model in the high-density limit. The obtained Hamiltonian H_{FB} has been compared with Sawada's Hamiltonian H_S, which also gives the exact ground state energy of the jellium model for $r_s \to 0$. In contrast with H_{FB} the Sawada Hamiltonian does not describe a fermion system. Consequently a full analysis of the high-density electron system in terms of H_S is questionable from a fundamental point of view. Notably the conclusion cannot be sustained that the elementary excitations of H_S do indeed correspond with the elementary excitations of the jellium model in the high-density limit.

The present boson formulation has not lead to new numerical results nor to a boson Hamiltonian that seems promising for a calculation of the properties of the electron system at lower densities. At first sight the appearing Hamiltonian H_B (2.17) seems an attractive starting-point. For it seems natural to neglect the very complicated term ΔH_B and to

diagonalize the bilinear part of $H_B^1+H_B^2$ while accounting for the remaining terms of $H_B^1+H_B^2$ by means of a perturbation calculation. However, there is a hidden problem here. Namely, at lower densities interaction terms with larger momentum transfers become important. This means that a neglect of ΔH_B can no longer be justified. Therefore such a perturbation procedure lacks a satisfactory systematics just as a perturbation calculation starting from Sawada's Hamiltonian, i.e. the results of such a calculation are just as questionable.

In summary: the significance of the present boson formulation lies in the fundamental sphere, not in the practical one. However, this does not mean that applications are excluded, e.g. a calculation of the electron–positron interactions as done previously by Arponen and Pajanne [9] using Sawada's method. For the present the question has been raised and answered in which way the interacting electron system can be described in terms of bosons.

Appendix A

The construction of the terms H_B^1 and H_B^2 of the boson Hamiltonian H_B (2.17) proceeds as follows. First H_B^1 is considered. In order to construct this term the following boson operator is introduced:

$$n_{k\sigma}^B = \theta(k_F - k) + \sum_{q\sigma'}{}' [d_{q\sigma}^+(k - q\sigma')d_{q\sigma}(k - q\sigma') - d_{q\sigma'}^+(k\sigma)d_{q\sigma'}(k\sigma)]. \tag{A.1}$$

It can be shown that:

(1) the boson states are eigenstates of $n_{k\sigma}^B$ and
(2) $n_{k\sigma}^B|\varphi_{Am}\rangle = n_{k\sigma}|m\rangle$.

Consequently it holds:

$$H_B^1 = \sum_{k\sigma}\left[\epsilon_k n_{k\sigma}^B - \frac{1}{2}\sum_q{}' V(q) n_{k+q\sigma}^B n_{k\sigma}^B\right]. \tag{A.2}$$

The term H_B^2 is obtained by expressing H_B^2 as

$$H_B^2 = \sum_{i=1}^{9} H_B^2(i) \tag{A.3}$$

where the nine terms $H_B^2(i)$ refer to the nine different types of off-diagonal matrix elements (2.8). These types are characterized by the signs of $|\ell_j| - k_F$, $j = 1, 2, 3, 4$, where it must be taken into account that a permutation of ℓ_1 and ℓ_2 or ℓ_3 and ℓ_4 does not lead to a new type of matrix element. The nine types are:

(1) $|\ell_1|, |\ell_2|, |\ell_3|, |\ell_4| > k_F$
(2) $|\ell_1|, |\ell_2|, |\ell_3|, |\ell_4| < k_F$
(3) $|\ell_1| < k_F, |\ell_2|, |\ell_3|, |\ell_4| > k_F$
(4) $|\ell_3| < k_F, |\ell_1|, |\ell_2|, |\ell_4| > k_F$
(5) $|\ell_1| > k_F, |\ell_2|, |\ell_3|, |\ell_4| < k_F$
(6) $|\ell_3| > k_F, |\ell_1|, |\ell_2|, |\ell_4| < k_F$
(7) $|\ell_1|, |\ell_2| < k_F, |\ell_3|, |\ell_4| > k_F$
(8) $|\ell_1|, |\ell_2| < k_F, |\ell_3|, |\ell_4| < k_F$
(9) $|\ell_1|, |\ell_4| < k_F, |\ell_2|, |\ell_3| > k_F$.

The construction of the terms $H_B^2(i), i = 1, \ldots, 9$ is straightforward. As an example the term $H_B^2(1)$ is constructed here. The translation of fermion states into corresponding boson states is facilitated by putting $\ell_1 = k_1 + q_1$, $\ell_2 = k_2 + q_2$, $\ell_3 = k_1 + q_1'$, $\ell_4 = k_1 + q_2'$ with $|k_1|, |k_2| < k_F$. Then the states $|m\rangle$ and $|m'\rangle$ as given by (2.4) and (2.7) can be expressed as follows:

$$\begin{aligned} |m\rangle &= c^+_{k_1+q_1\tau_1} c_{k_1\sigma_1} c^+_{k_2+q_2\tau_2} c_{k_2\sigma_2} |m-2\rangle \\ |m'\rangle &= c^+_{k_1+q_1'\tau_1'} c_{k_1\sigma_1} c^+_{k_2+q_2'\tau_2'} c_{k_2\sigma_2} |m-2\rangle \end{aligned} \tag{A.4}$$

where the fermion operators refer to six mutually different fermion states and $|m-2\rangle$ is given by

$$|m-2\rangle = \left(\prod_{i=3}^{m} \theta(k_F - |k_i|)\theta(|k_i + q_i| - k_F) c^+_{k_i+q_i\tau_i} c_{k_i\sigma_i} \right) |0\rangle. \tag{A.5}$$

Clearly the corresponding boson states $|\varphi_{Am}\rangle$ and $|\varphi_{Am'}\rangle$ read:

$$\begin{aligned} |\varphi_{Am}\rangle &= d^+_{q_1\tau_1}(k_1\sigma_1) d^+_{q_2\tau_2}(k_2\sigma_2) |\varphi_{Am-2}\rangle \\ |\varphi_{Am'}\rangle &= d^+_{q_1'\tau_1'}(k_1\sigma_1) d^+_{q_2'\tau_2'}(k_2\sigma_2) |\varphi_{Am-2}\rangle \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} d_{q_1\tau_1}(k_1\sigma_1)|\varphi_{Am-2}\rangle = d_{q_2\tau_2}(k_2\sigma_2)|\varphi_{Am-2}\rangle = d_{q_1'\tau_1'}(k_1\sigma_1)|\varphi_{Am-2}\rangle \\ = d_{q_2'\tau_2'}(k_2\sigma_2)|\varphi_{Am-2}\rangle = 0. \end{aligned} \tag{A.7}$$

The requirement (A.7) follows directly from the correspondence of $|\varphi_{Am-2}\rangle$ with the fermion state $|m-2\rangle$ given by (A.5). Using

$$\begin{aligned} \langle\varphi_{Am}|H_B^2(1)|\varphi_{Am'}\rangle = [V(q_1 - q_1')\delta_{\tau_1\tau_1'}\delta_{\tau_2\tau_2'} \\ - V(k_2 + q_2 - k_1 - q_1')\delta_{\tau_2\tau_1'}\delta_{\tau_1\tau_2'}]\delta_{q_1+q_2,q_1'+q_2'} \end{aligned} \tag{A.8}$$

for any pair of boson states of the type given by (A.6), $H_B^2(1)$ is found to have the form

$$\begin{aligned} H_B^2(1) = \frac{1}{2} \sum_{p_1,p_2,p_1',p_2',\tau,\tau'}{}' \sum_{k,k',\sigma,\sigma'}{}' \delta_{p_1+p_2,p_1'+p_2'} \\ \times [V(p_1 - p_1') d^+_{p_1\tau}(k\sigma) d^+_{p_2\tau'}(k'\sigma') d_{p_1'\tau}(k\sigma) d_{p_2'\tau'}(k'\sigma') \\ - V(k' + p_2 - k - p_1') d^+_{p_1'\tau}(k\sigma) d^+_{p_2\tau'}(k'\sigma') d_{p_1'\tau'}(k\sigma) d_{p_2'\tau}(k'\sigma')] \end{aligned} \tag{A.9}$$

where the primes appearing in the summations indicate that terms containing $V(0)$ are excluded as well as that the boson operators must be such that the corresponding fermion operators are all different. Note further that the relevant matrix elements of $H_B^2(1)$ between the boson states of subset A are indeed all zero except for those pairs of states as given by (A.6). Consequently $H_B^2(1)$ is correctly expressed by (A.9).

The remaining terms of $H_B^2(1)$ (A.3) can be constructed analogous to $H_B^2(1)$. They are found to be

$$\begin{aligned} H_B^2(2) = \frac{1}{2} \sum_{k_1,k_2,k_1',k_2',\tau,\tau'}{}' \sum_{q,q',\sigma,\sigma'}{}' \delta_{k_1+k_2,k_1'+k_2'} \\ \times [V(k_1 - k_1') d^+_{q+k_1'-k_1\sigma}(k_1\tau) d^+_{q'+k_2'-k_2\sigma'}(k_2\tau') d_{q\sigma}(k_1'\tau) d_{q'\sigma'}(k_2'\tau') \\ - V(k_2 - k_1') d^+_{q+k_1'-k_1\sigma}(k_1\tau) d^+_{q'+k_2'-k_2\sigma'}(k_2\tau') d_{q\sigma}(k_1'\tau') d_{q'\sigma'}(k_2'\tau)] \end{aligned} \tag{A.10}$$

$$H_B^2(3) = \sum_{q,p,p',\tau}{}' \sum_{k,k',\sigma,\sigma'}{}' [V(q)d^+_{p'\tau}(k'\sigma')d^+_{q\sigma}(k\sigma)d_{p\tau}(k'\sigma') - V(p'+k'-k)d^+_{p'\sigma}(k'\sigma')d^+_{q\tau}(k\sigma)d_{p\tau}(k'\sigma')]\delta_{p,q+p'} \tag{A.11}$$

$$H_B^2(4) = [H_B^2(3)]^+ \tag{A.12}$$

$$H_B^2(5) = \sum_{k_1,k_2,k_1',\tau}{}' \sum_{q,q',\sigma,\sigma'}{}' [V(q')d^+_{q+k_1'-k_2\sigma}(k_2\tau)d_{q\sigma}(k_1'\tau)d_{q'\sigma'}(k_1\sigma') - V(k_2-k_1)d^+_{q+k_1'-k_2\sigma}(k_2\tau)d_{q\sigma}(k_1'\sigma')d_{q'\sigma'}(k,\tau)]\delta_{q'+k_2,k_1'} \tag{A.13}$$

$$H_B^2(6) = [H_B^2(5)]^+ \tag{A.14}$$

$$H_B^2(7) = \frac{1}{2}\sum_{q,k,k',\sigma,\sigma'}{}' [V(q)d^+_{-q\sigma'}(-k'\sigma')d^+_{q\sigma}(k\sigma) - V(q+k+k')d^+_{-q\sigma'}(-k'\sigma)d^+_{q\sigma}(k\sigma')] \tag{A.15}$$

$$H_B^2(8) = [H_B^2(7)]^+ \tag{A.16}$$

$$H_B^2(9) = \sum_{q,k,k'\sigma,\sigma'}{}' [V(q)d^+_{q\sigma'}(k'\sigma')d_{q\sigma}(k\sigma) - V(k-k')d^+_{q\sigma}(k'\sigma')d_{q\sigma}(k\sigma')]. \tag{A.17}$$

The primes appearing in the summations indicate that the terms with $V(0)$ are excluded as well as that the appearing boson operators must be such that the corresponding fermion operators are all different.

Appendix B

Consider the Hamiltonian $\tilde{H}_F'$, which is obtained from $\tilde{H}_F$ (3.1) by taking into account only those interaction terms with momentum transfer $q < q_0 << k_F$, where q_0 is some cut-off momentum transfer. In the high-density limit ($r_s \to 0$), the ground state energy of $\tilde{H}_F'$ can be expressed as follows [7, 8]:

$$E_0 = 2N\left[\frac{2.2}{r_s^2} - \frac{0.9}{r_s} + \frac{3}{16\pi^2 r_s^2}\left(\frac{9\pi}{4}\right)^{2/3}\int_{-\infty}^{\infty}\mathrm{d}u\int_0^{q_0/k_F} 4\pi q^3\mathrm{d}q \times\left[\ln\left[1+\frac{\chi(u)}{q^2}\right]-\frac{\chi(u)}{q^2}\right]\right] \tag{B.1}$$

where

$$\chi(u) = (4r_s/\pi)(4/9\pi)^{1/3}[1 - u\arctan(1/u)]. \tag{B.2}$$

The integral over q can be easily calculated leading to

$$E_0 = 2N\left[\frac{2.2}{r_s^2} - \frac{0.9}{r_s} + \frac{3}{16\pi r_s^2}\left(\frac{9\pi}{4}\right)^{2/3}\int_{-\infty}^{\infty}\mathrm{d}u\left[\left(\frac{q_0^4}{k_F^4}-\chi^2\right)\times\ln\left(\chi+\frac{q_0^2}{k_F^2}\right)-\chi\frac{q_0^2}{k_F^2}-\frac{q_0^4}{k_F^4}\ln\frac{q_0^2}{k_F^2}+\chi^2\ln\chi\right]\right] \tag{B.3}$$

This result gives the exact ground state energy (3.2) in the limit $r_s \to 0$ provided that $q_0^2/k_F^2 \gg \chi_{max}$, where $\chi_{max} = \chi(0) = (4r_s/\pi)(4/9\pi)^{1/3}$ is the maximum value of the function χ. This requirement is satisfied by choosing $q_0 = r_s^{1/4}k_F$ as χ_{max} is proportional to r_s.

In order to show that H_F (3.5) leads to the exact ground state energy (3.2) it must be argued now that the additional restriction $|q+k+k'| > r_s^{1/4}k_F$, which is the only difference between H_F and $\tilde{H}_F'$, does not affect E_0 up to order $\ln r_s$, i.e.

$$\lim_{r_s\to 0}\left[\frac{(1/2N)[\langle\varphi_0|\tilde{H}_F'|\varphi_0\rangle - \langle\varphi_0|H_F|\varphi_0\rangle]}{\ln r_s}\right] = 0 \tag{B.4}$$

where $|\varphi_0\rangle$ is the ground state of $\tilde{H}_F'$. For that purpose the following function is introduced:

$$F_{\sigma\sigma'}(q,k,k') = \tfrac{1}{2}\theta(r_s^{1/4}k_F - q)\theta(k_F - k)\theta(k_F - k')\theta(|k+q| - k_F).\theta(|k'+q| - k_F)V(q)$$
$$\times\left[\langle\varphi_0|[c^+_{-k'-q\sigma'}c_{-k'\sigma'} + c^+_{k'\sigma'}c_{k'+q\sigma'}][c^+_{k+q\sigma}c_{k\sigma} + c^+_{-k\sigma}c_{-k-q\sigma}]|\varphi_0\rangle\right]. \tag{B.5}$$

Using (3.1) together with the definition of $\tilde{H}_F'$ and using (3.2) and (3.5) it appears that in the limit $r_s \to 0$:

$$\sum_{k,k',q,\sigma,\sigma'}{}' F_{\sigma\sigma'}(q,k,k') = 2N(0.0622\ln r_s)$$
$$\sum_{k,k',q,\sigma,\sigma'}{}' \theta(r_s^{1/4}k_F - |k+k'+q|)F_{\sigma\sigma'}(q,k,k') = \langle\varphi_0|\tilde{H}_F'|\varphi_0\rangle - \langle\varphi_0|H_F|\varphi_0\rangle. \tag{B.6}$$

Consequently $\tilde{H}_F'$ and H_F have the same ground state energy in the high-density limit if

$$\lim_{r_s\to 0}\left[\left(\sum_{k,k',q,\sigma,\sigma'}{}' \theta(r_s^{1/4}k_F - |k+k'+q|)F_{\sigma\sigma'}(q,k,k')\right)\Big/\left(\sum_{k,k',q,\sigma,\sigma'}{}' F_{\sigma\sigma'}(q,k,k')\right)\right] = 0. \tag{B.7}$$

The denominator in the left-hand side of (B.7) can be expressed as follows in the limit $r_s \to 0$:

$$\sum_{k,k',q,\sigma,\sigma'}{}' F_{\sigma\sigma'}(q,k,k') = C\int_0^{q_0} q^2\mathrm{d}q\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\left[\int_{k_F-qx}^{k_F} k^2\mathrm{d}k\right.$$
$$\left.\times\int_{k_F-qy}^{k_F} k'^2\mathrm{d}k'\int_0^{2\pi}\mathrm{d}\varphi F_{\sigma\sigma'}(q,k,k',x,y)\right] \tag{B.8}$$

where x, y and φ are defined by

$$k\cdot q = kqx$$
$$k'\cdot q = k'qy$$
$$k\cdot k' = kk'[xy - \sqrt{1-x^2}\sqrt{1-y^2}\cos\varphi] \tag{B.9}$$

$q_0 = k_F r_s^{1/4}$ and C is some constant. It should be remarked that the function F is independent of φ. The reason being that each function value $F_{\sigma\sigma'}(q,k,k')$ can be interpreted as a sum

over ring diagrams. The numerator in (B.7) is also of the form (B.8), but now the following restriction must be imposed:

$$q^2 + k^2 + k'^2 + 2qkx + 2qk'y + 2kk'[xy - \sqrt{1-x^2}\sqrt{1-y^2}\cos\varphi] < k_F^2 r_s^{1/2}. \tag{B.10}$$

It can easily be checked that for $r_s \to 0$, where it holds that $k - k_F$, $k' - k_F$ and q are of the order of $r_s^{1/4}k_F$ or smaller, the restriction (B.10) leads to:

$$1 - \tfrac{1}{2}\varepsilon^2 \leqslant \cos\varphi \leqslant 1 \tag{B.11}$$

i.e.

$$0 \leqslant \varphi \leqslant \varepsilon \tag{B.12}$$

where ε is of the order of $r_s^{1/4}$. From (B.8) and (B.12) it follows directly that the requirement (B.7) is satisfied, i.e. $\bar{H}_F'$ and H_F have the same ground state energy in the high-density limit.

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